Linear system

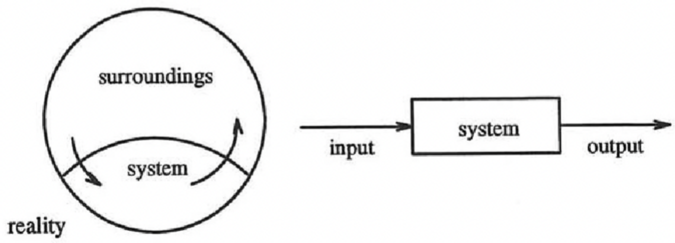
Zambelli Lorenzo
BSc Applied Mathematics

April 2021-June 2021

What is system theory?

System theory is the mathematical theory of dynamical system and feedback control.

A system is part of reality which we think to be a separated unit within this reality. The reality outside the system is called the surroundings. The interaction between system and surroundings is realized via quantities, quite of ten functions of time, which are called input and output. The system is influenced via the input(-functions) and the system has an influence on the surroundings by means of the output(-functions).



Mathematical system(s) theory is concerned with the study and control of input / output phenomena. There is no difference between the terminologies 'system theory' and 'systems theory' both are used in the (scientific) literature and will be used interchangeably. The emphasis in system(s) theory is on the dynamic behaviour of these phenomena, i.e. how do characteristic features (such as input and output) change in time and what are the relationships, also as functions of time.

Ordinary Differential equations

Scalar first-order equations

Definition 1

$$\dot{x}(t) = f(t, x(t)) \qquad f : D \to \mathbb{R} \qquad D \subset \mathbb{R} \times \mathbb{R} \tag{1}$$

with

$$\begin{aligned} t &: \quad \textit{independent variable} \\ x &: \quad \textit{dependent variable} \\ \dot{x}(t) &= \frac{d}{dt}x(t) \end{aligned}$$

Definition 2 *A solution to (1) is a function* $x : J \to \mathbb{R}$ *if*

1. *x is differentiable*
2. *$(t, x(t)) \in D$ for all $t \in J$*
3. *$\dot{x}(t) = f(t, x(t))$*

Remark: The scalar differential equation has a simple geometric interpretation. Namely, if x is a solution that passes through (t_0, x_0) (in this case, $x_0 = x(t_0)$), then $f(t_0, x_0)$ gives the slope of the curve x at this point. Following this interpretation, the slope $f(t, x)$ can be indicated for each $(t, x) \in D$. This leads to a so-called **direction field**. A solution to the differential equation can then be regarded as a curve that fits the direction field.

Definition 3 ***Initial value problem*** *Given $\dot{x}(t) = f(t, x(t))$, $(t_0, x_0) \in D$ find a solution that satisfies the initial condition $x(t_0) = x_0$*

Definition 4 *Equation of the form $\dot{x}(t) = f(t)$ Let $\dot{x}(t) = f(t)$ be continuous in a given interval, then by the fundamental theorem of calculus*

$$\begin{aligned} x(t) &= \int f(t)dt + c \\ &= F(t) + c \end{aligned}$$

with $\frac{dF}{dt}(t) = f(t)$

Definition 5 ***Equations of the form*** *$\dot{x}(t) = g(x)$* ***autonomus*** *Let $\dot{x}(t) = g(x(t))$ be continuous in a given interval, the it follows the following Property:*

for $g(x) \neq 0$ any solution to (2) is of the form $H(x) = t + c$ for any $\tau \in \mathbb{R}$ with $\frac{dH}{dx}(1) = \frac{1}{g(x)}$

Definition 6 ***Separeble equations*** *Let $\dot{x}(t) = f(t)g(t)$ be continuous in a given interval, then for $g(x) \neq 0$, any solution is of the form*

$$H(x) = F(t) + c \quad \textit{with } \frac{dH}{dx}(x) = \frac{1}{g(x)}, \ \frac{dF}{dt}(t) = f(t)$$

Remark: Remeber the separations of variables method:

$$\begin{aligned} \frac{dx}{dt} = f(t)g(x) \rightarrow \int \frac{1}{g(x)}dx &= \int f(t)dt \\ \Leftrightarrow H(x) &= F(t) + c \end{aligned}$$

Linear scalar differential equations

A linear differential equation is a equation of the form

$$\dot{x}(t) = a(t)x(t) + b(t) \quad \text{with } a, b : J \to \mathbb{R} \text{ continuous}$$

The equation is said:

- **Homogeneous** if $b(t) = 0$ for all t

- **Nonhomogeneous** if $b(t) \neq 0$ for all t

Remark: Roughly speaking, the differential equation is called linear because the terms related to the dependent variable x (namely, both x and $\dot{x}(t)$) only appear linearly. A more particular one is to use the definition of linearity with the differetial operator, i.e. $l(x) = \dot{x}(t) - a(t)x$

Theorem 7 ***Solution to a homogenuos equation*** *Consider the initial value problem*

$$\dot{x}(t) = a(t)x(t), \quad x(t_0) = x_0$$

where $a : J \to \mathbb{R}$ is continuous and $t_0 \in J$. Then the unique solution is

$$x(t; t_0, x_0) = x_0 e^{F(t)}, \quad F(t) = \int_{t_0}^{t} a(\tau)\, d\tau$$

for $t \in J$

Remark: Note that we can arrive at this solution by used of both separation variable method or the integrating factor method

Theorem 8 ***Solution to a nonhomogenuos equation*** *Consider the initial value problem*

$$\dot{x}(t) = a(t)x(t) + b(t), \quad x(t_0) = x_0$$

where $a : J \to \mathbb{R}$ is continuous and $t_0 \in J$. Then the unique solution is

$$x(t; t_0, x_0) = x_0 e^{F(t)} + e^{F(t)} \int_{t_0}^{t} e^{-F(\tau)} b(\tau)\, d\tau, \quad F(t) = \int_{t_0}^{t} a(\tau) d\tau$$

for $t \in J$

Remark: To solve this we can use the variation of costant method, i.e if $x(t) = ce^{F(t)}$ is the solution of the homogeneous equation, then by vary the constant we obtain $x(t) = z(t)e^{F(t)}$ and $z(t) = e^{-F(t)}x(t) \Rightarrow \dot{z}(t) = e^{-F(t)}b(t)$ form which we get that $z(t) = \int e^{-F(t)}b(t)\, dt + c$. Then we substitute $z(t)$ into the solution of the homogeneous equation, then we are done.

We can also use the homogeneous + particuar method, where the general solution can be written as $x(t) = x_h(t) + x_p(t)$, here follows the method:

1. Solve the homogeneous equation
2. Find one particular solution to the nonhomogenous equation using Variation of Constant.
3. Find the general solution

Linear system

A linear system is a system of the following equations:

$$\dot{x}(t)_1 = f_1(t, x_1(t), x_2(t), ..., x_n(t))$$
$$\vdots$$
$$\dot{x}(t)_n = f_n(t, x_1(t), x_2(t), ..., x_n(t))$$

In compact form we have $\dot{x}(t) = f(t, x(t))$ with $f : D \to \mathbb{R}^n$, where

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; f(t,x) = \begin{bmatrix} f_1(t, x_1(t), x_2(t), ..., x_n(t)) \\ \vdots \\ f_n(t, x_1(t), x_2(t), ..., x_n(t)) \end{bmatrix}$$

One of the motivations for studying systems of differential equations is that they can be used to represent higher-order differential equations. To illustrate this, denote

$$y^{(k)}(t) = \frac{d^k y}{dt^k}(t), \quad y^n(t) = f(t, y(t), \dot{y}(t), ..., y^{(n-1)}(t))$$

Then if we introduce

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

it yields,

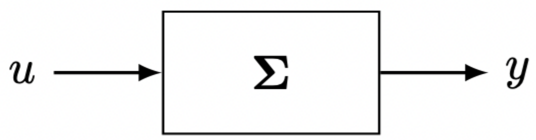
$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(t, x_1(t), x_2(t), ..., x_{n-1}(t)) \end{bmatrix}$$

Definition 9 ***Linear system in state-space form*** *Let $u : J \to \mathbb{R}^m$ be the input function and $y : J \to \mathbb{R}^p$ be the output function, then the linear system is denoted by*

$$\Sigma : \begin{cases} \dot{x}(t) & = Ax(t) + Bu(t) = f(t,x) \\ y(t) & = Cx(t) + Du(t) \end{cases}$$

with $x(t) \in \mathbb{R}^n$ denoted as ***the state function****.*

A linear system Σ is often depicted as a simple block diagram, clearly indicating that a system



relates inputs and outputs.

Definition 10 ***Non linear systems***

$$\Sigma_{nl} : \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$

Definition 11 ***Equilibrium*** *Let* $u(t) = \overline{u}$ *be constant. Then,* $\overline{x} \in \mathbb{R}^n$ *is a equilibrium for* $\overline{u}$ *if*

$$f(\overline{x}, \overline{u}) = 0$$

It means that $x(t) = \overline{x}$*, i.e. constant solution, then* $\dot{x}(t) = f(\overline{x}, \overline{u}) = 0$

Remark: in practice the equilibrium point is a point where you stay there for ever.

Definition 12 ***Linearization*** *Let* $(\overline{x}, \overline{u})$ *be an equilibrium of non linear system* Σ_{nl}*. Then the linear system:*

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + B\tilde{u}(t) \\ \tilde{y}(t) &= C\tilde{x}(t) + D\tilde{u}(t) \end{aligned}$$

with state $\tilde{x} \in \mathbb{R}^n$*, input* $\tilde{u} \in \mathbb{R}^m$*, output* $\tilde{y} \in \mathbb{R}^p$ *and*

$$A = \frac{\partial f}{\partial x}(\overline{x}, \overline{u}) \; B = \frac{\partial f}{\partial u}(\overline{x}, \overline{u}) \; C = \frac{\partial h}{\partial x}(\overline{x}, \overline{u}) \; D = \frac{\partial h}{\partial u}(\overline{x}, \overline{u})$$

where A,B,C,D are Jacobian matrix.

Remark: $\tilde{y}(t) = y(t) - \overline{y}$ and $\tilde{x}(t) = x(t) - \overline{x}$ $\tilde{u} = u(t) - \overline{u}$

Solutions of homogeneous linear systems

Initial value problem

$$\dot{x}(t) = Ax(t), \quad x(t_0) = x_0 \tag{2}$$

i.e., find a differentiable function $x : J \to \mathbb{R}^n$ s.t. (1)

Theorem 13 *Consider the initial valued problem with* $A \in \mathbb{R}^{n\times n}$*. The function*

$$x(t; t_0, x_0) = e^{A(t-t_0)}x_0$$

$t \in \mathbb{R}$*, is the unique solution to this initial value problem.*

Remark: we have reach this theorem by first using the method of successive approximations from the solution of the initial valued problem $x(t) = x_0 + \int_{t_0}^{t} Ax(\tau)\, d\tau$, from which we get

$$x^{(k)}(t) = \left(\sum_{l=0}^{k} \frac{A^l (t-t_0)^l}{l!} \right) x_0, \quad A^0 = I$$

where $x^{(k)}$ means the k approximation of $x(t)$. Then we use the definition of matrix exponential.

Solutions of nonhomogeneuos linear systems

Initial valued problem

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \tag{3}$$

and $u : J \to \mathbb{R}^m$ given, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$

Theorem 14 *The unique solution to IVP is given as*

$$x(t; t_0, x_0, u) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau$$

Output solution

Given $y(t) = Cx(t) + Du(t)$ and the initial valued problem

$$y(t; t_0, x_0, u) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\, d\tau + Du(t)$$

Linearity & time invariance

Theorem 15 *Let* (t_0, x_0) *and* (t_0, x_0') *and* $u, u' : J \to \mathbb{R}^m$*. Then,*

$$x(t; t_0, \alpha x_0 + \alpha' x_0', \alpha u + \alpha' u') = \alpha x(t; t_0, x_0, u) + \alpha' x(t; t_0, x_0, u)$$

for all $\alpha, \alpha' \in \mathbb{R}$

Remark: $x(t; t_0, x_0, u) = x(t; t_0, x_0, 0) + x(t; t_0, 0, u)$

Theorem 16 *For any* (t_0, x_0) *and* $u : \mathbb{R} \to \mathbb{R}^m$*,*

$$x(t; t_0, x_0, u) = x(t - t_0; 0, x_0, u_{t_0})$$

with $u_{t_0}(t) = u(t + t_0)$*,* $t \in \mathbb{R}$

Remark: Time invariance means that whether we apply an input to the system now or T seconds from now, the output will be identical, i.e. the system is time invariant because the output does not depend on the particular time the input is applied.

Computation of the matrix exponential

Definition 17 *For* $A \in \mathbb{R}^{n\times n}$ *(*$A \in \mathbb{C}^{n\times n}$*)*

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + \frac{1}{2}A^2t^2 + \cdots$$

Lemma 18 *Let* $T \in \mathbb{C}^{n\times n}$ *be nonsigular and* $A \in \mathbb{C}^{n\times n}$*. Then,*

$$e^{TAT-1t} = Te^{At}T^{-1} \quad \textit{for all } t$$

Theorem 19 *Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable. Then,*

$$x(t;t_0,x_0) = e^{A(t-t_0)}x_0 = \sum_{i=0}^{n} c_i v_i e^{\lambda_i(t-t_0)}$$

for some constants c_i and (λ_i, v_i) eigenpairs of A. Where $Tc = x_0$.

Lemma 20 *The following holds for any $A \in \mathbb{C}^{n\times n}$,*

1. $(e^{At})^{-1} = e^{-At}$
2. $\frac{d}{dt}\{e^{At}\} = Ae^{At} = e^{At}A$
3. $e^{At}B = Be^{At}$ *if A and B commute*

Lemma 21

1. *Let $A, B \in \mathbb{C}^{n\times n}$ commute, Then*
$$e^{At}e^{Bt} = e^{(A+B)t}$$
2. $e^{At}e^{As} = e^{A(t+s)}$ *for all $t, s \in \mathbb{R}$*

Definition 22 ***The Jordan canonical form*** *A Jordan block ($J_k(\lambda) \in \mathbb{C}^{k\times k}$) is the matrix*

$$J_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & & \\ & \lambda_k & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_k \end{bmatrix}$$

where $J_k(\lambda) = \lambda I + N$ with $N^k = 0$

Lemma 23

$$e^{J_k(\lambda)t} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k-2}}{(k-2)!} & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \frac{t^2}{2!} \\ & & & & \ddots & t \\ & & & & & 1 \end{bmatrix}$$

Theorem 24 *For any $A \in \mathbb{R}^{n\times n}$, there exits a nonsigular $T \in \mathbb{C}^{n\times n}$ such that $A = TJT^{-1}$ with*

$$J = \begin{bmatrix} J_{k_1}(\lambda_1) & & & \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{k_r}(\lambda_n) \end{bmatrix}$$

with $\lambda_i \in \sigma(A)$, where $\sigma(A) = \{\lambda | Av = \lambda v \text{ for } v \neq 0\}$ is the spectrum. Conversely, if $\lambda \in \sigma(A)$, then $\lambda = \lambda_i$ for some $i \in \{1, 2, ..., r\}$ and $n = k_1 + \cdots + k_r$

Computing the matrix exponetial e^{At}

1. Compute the Jordan canonical form $A = TJT^{-1}$
2. For each Jordan block $J_{k_i}(\lambda_i)$, compute $e^{J_{k_i}(\lambda_i)t}$
3. Compute e^{At} using

$$\begin{aligned} e^{At} = e^{TJT^{-1}t} &= Te^{Jt}T^{-1} \\ &= T \begin{bmatrix} e^{J_{k_1}(\lambda_1)t} & & \\ & \ddots & \\ & & e^{J_{k_r}(\lambda_r)t} \end{bmatrix} T^{-1} \end{aligned}$$

Stability

Definition 25 *Let consider a system without input, i.e. $\dot{x}(t) = Ax(t)$, this system is called*

1. *stable if each $x_0 \in \mathbb{R}^n$, there exists $M > 0$ s.t.*

$$|x(t; x_0)| = |e^{At}x_0| \leq M \quad \text{for all } t \geq 0$$

2. *asymptotically stable if, for each $x_0 \in \mathbb{R}^n$*

$$\lim_{t\to\infty} x(t; x_0) = \lim_{t\to\infty} e^{At}x_0 = 0$$

where $x(t; x_0) = e^{At}x_0$

Remark: the stability of a system is define on the system without input.
Observations:

1. Denote by $(e^{At})_{ij}$ the element i, j in e^{At}

$$\begin{aligned} &|(e^{At})_{ij}| \leq m \quad \text{for all } i, j \Rightarrow \dot{x}(t) = Ax(t) \text{ is stable} \\ &\lim_{t\to\infty} (e^{At})_{ij} = 0 \text{ for all } i, j \Rightarrow \dot{x}(t) = Ax(t) \text{ is asymptotically stable} \end{aligned}$$

2. Each $(e^{At})_{ij}$ is a sum of terms of the form $t^k e^{\lambda t}$ with k non negative integer and λ is an eigenvalue of A

Lemma 26 *Consider $t \mapsto t^k e^{\lambda t}$ with $k \geq 0$,*

1. *if $\Re(\lambda) < 0$, then $\lim_{t\to\infty} t^k e^{\lambda t} = 0$ and $|t^k e^{\lambda t}| \leq M$ for all $t \geq 0$*
2. *For any $\alpha \in \mathbb{R}$ such that $\Re(\lambda) < \alpha$,*

$$|t^k e^{\lambda t}| \leq Me^{\alpha t} \quad \text{for all } t \geq 0$$

Denote

$$\begin{aligned} \mathbb{C}_- &= \{z \in \mathbb{C} \,|\, \Re(z) < 0\} \\ \overline{\mathbb{C}}_- &= \{z \in \mathbb{C} \,|\, \Re(z) \leq 0\} \end{aligned}$$

Theorem 27 *The system $\dot{x}(t) = Ax(t)$ is*

1. *Stable if only if $\sigma(A) \subset \overline{\mathbb{C}}_-$ and every $\lambda \in \sigma(A)$ with $\Re(\lambda) = 0$ is semisimple ($a\lambda = g\lambda$)*
2. *asymptotically stable if and only if $\sigma(A) \subset \mathbb{C}_-$. In this case, $\exists M, \gamma > 0$ s.t. $\|e^{At}\| \leq Me^{-\gamma t}$, $t \geq 0$*

Routh-Harwitz criterion

Definition 28 *A polynomial*

$$p(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{4}$$

with $a_i \in \mathbb{R}$, $a_n \neq 0$ *is called stable if all its roots* ($p(\lambda) = 0$ *have negative real part :* $\sigma(p) \subset \mathbb{C}_-$

Theorem 29 ***Routh-Harwitz*** *The polynomial (4) is stable if and only if*

1. $a_{n-1} \neq 0$ *and has the same sign as* a_n*, i.e,* $a_n a_{n-1} > 0$
2. *the polynomial*

$$q(s) = a_{n-1}p(s) - a_n(a_{n-1}s^n + a_{n-3}s^{n-2} + a_{n-5}s^{n-4} + ...)$$

is stable

Lemma 30 *Let* p *with* $a_i \in \mathbb{R}$ *be stable. Then, all its coefficients are nonzero and have the same sign*

Interval polynomials

Let $a_i^-, a_i^+ \in \mathbb{R}^n$, $i = 1, ..., n$ satisfy $a_i^- \leq a_i^+$ and define the set of polynomials $\mathcal{P}$ as

$$\mathcal{P}(s) = \left\{ a_n s^n + a_{n-1}s^n + \cdots + a_1 s + a_0 \,|\, a_i^- \leq a_i \leq a_i^+ \text{ for all } i \in \{0, 1, ..., n\} \right\} \tag{5}$$

Then, the stability of the set $\mathcal{P}$ is defined as follows

Definition 31 *The set of polynomials* $\mathcal{P}$ *as in* (5) *is called stable if each polynomial in the set is stable, i.e., if* p *is stable for all* $p \in \mathcal{P}$

Theorem 32 ***Kharitonov's theorem*** *The set of polynomials* $\mathcal{P}$ *as in* (5) *is stable if and only if the following four polynomials are all stable:*

$$\begin{aligned}
p^{++}(s) &= a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_4^+ s^4 + a_5^+ s^5 + a_6^- s^6 + \cdots, \\
p^{+-}(s) &= a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^+ s^4 + a_5^- s^5 + a_6^- s^6 + \cdots, \\
p^{-+}(s) &= a_0^- + a_1^+ s + a_2^+ s^2 + a_3^- s^3 + a_4^- s^4 + a_5^+ s^5 + a_6^+ s^6 + \cdots, \\
p^{--}(s) &= a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + a_4^- s^4 + a_5^- s^5 + a_6^+ s^6 + \cdots
\end{aligned}$$

Remark: If a coefficient a_i is fixed, we just set the lower and upper bound to be equal, i.e., $a_i^- = a_i^+$.
The importance of Kharitonov's theorem follows from its simplicity. Essentially, it reduces checking stability of infinitely many polynomials to checking stability of only four representative polynomials. For the latter, the Routh-Hurwitz criterion can be used.

Controllability

Controllability is related to the question to what extend the state trajectories of a linear system can be influenced through the input.

Let the system

$$\Sigma : \quad \dot{x}(t) = Ax(t) + Bu(t) \tag{6}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$

Definition 33 *The state $x_f \in \mathbb{R}^n$ is reachable at time $T > 0$ if there exists $u : [0, T] \to \mathbb{R}^m$ such that*

$$x(T; 0, u) = x_f$$

Definition 34 ***Reachable subspace*** W_T

$$\begin{aligned} W_T &= \{x_f \in \mathbb{R}^n \,|\, x_f \textit{ is reachable at } T\} \\ &= \left\{ \int_0^T e^{A(T-\tau)} Bu(\tau)\, d\tau \,|\, u : [0, T] \to \mathbb{R}^m \right\} \end{aligned}$$

Definition 35 *The system Σ is reachable at time $T > 0$ if any $x_f \in \mathbb{R}^n$ is reachable at time $T > 0$, i.e, $W_T = \mathbb{R}^n$*

Theorem 36 *Let $v \in \mathbb{R}^n$ and $T > 0$. Then, the following are equivalent:*

1. *$v^T x = 0$ for all $x \in W_T$ (i.e v is orthogonal to W_T)*
2. *$v^T e^{At} B = 0$ for all $0 \leq t \leq T$*
3. *$v^T A^k B = 0$ for $k = 0, 1, 2, ...$*
4. *$v^T [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = 0$*

Corollary 37 *W_T is independent of T and $W_T = \operatorname{im}[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$*

Definition 38 *The system Σ is controllable at time $T > 0$ if, for any $x_0, x_f \in \mathbb{R}^n$, there exists $u : [0, T] \to \mathbb{R}^m$ such that*

$$x(T; x_0, u) = x_f$$

Theorem 39 *The system Σ is controllable at $T > 0$ if and only if Σ is reachable at $T > 0$*

Theorem 40 *The following are equivalent*

1. *$\exists T > 0$ such that Σ is controllable at time T*
2. *Σ is controllable at T for all $T > 0$*
3. *$\operatorname{rank}[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] = n$*
4. *$W = \mathbb{R}^n$*

Theorem 41 *The reachable subspace W is the smallest A-invariant subspace containing* $\operatorname{im} B$

Observability

Observability deals with the question to what extend can observer what happening in my system by looking only the output y, i.e it asks for the extend at which the state trajectories influence the output.

Definition 42 *Two states $x_0, x_0' \in \mathbb{R}^n$ are indistinguishable on $[0, T]$ if*

$$y(t; x_0, 0) = y'(t; x_0', 0) \quad \textit{for all} t \in [0, T]$$

Remark: $y(t; x_0, 0) = e^{At}x_0$

Definition 43 ***Unobservable subspace***

$$\begin{aligned} N_T &= \{x \in \mathbb{R}^n \,|\, x \;\; \textit{is indistinguishable from 0 on } [0, T] \\ &= \{x \in \mathbb{R}^n \,|\, Ce^{At}x = 0 \;\; \textit{for all } t \in [0, T]\} \end{aligned}$$

Theorem 44 *Let $T > 0$. Then, the following are equivalent*

1. $x \in N_T$
2. $Ce^{At}x = 0$ *for all* $0 \leq t \leq T$
3. $CA^k x = 0$ *for all* $k = 0, 1, 2, ...$
4. $\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x = 0$

Corollary 45 *N_T is independent of T for $T > 0$ and*

$$N_T = \ker \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Remark: $Let M \in \mathbb{R}^{n \times n}$ then $\ker(M) = \{x \in \mathbb{R}^n \,|\, Mx = 0\}$

Definition 46 *The system Σ is observable on $[0, T]$ if $x_0, x_0' \in \mathbb{R}^n$ are indistinguishable on $[0, T]$ only if $x_0 = x_0'$*

Theorem 47 *The unobservable subspace N is the largest A-invariant subspace contained in* $\ker C$

Theorem 48 *The following are equivalent*

1. $\exists T > 0$ *such that Σ is observable on* $[0, T]$

2. Σ *is observable on* $[0,T]$ *for all* $T>0$

3. $\operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

4. $N=\{0\}$

Similar systems

Definition 49 *$\Sigma(A,B)$ and $\Sigma(\overline{A},\overline{B})$ are called similar if there exists a nonsingular $T\in\mathbb{R}^{n\times n}$ such that*

$$\overline{A}=TAT^{-1}, \quad \overline{B}=TB$$

Theorem 50 *Let $\Sigma(A,B)$ and $\Sigma(\overline{A},\overline{B})$ be similar, then $\exists T\in\mathbb{R}^{n\times n}$ be non singular, then*

$$\begin{aligned} \Sigma(A,B) \ \textit{ is controllable } &\Leftrightarrow \Sigma(TAT^{-1},TB) \ \textit{ is contrallable} \\ \Sigma(A,C) \ \textit{ is observable } &\Leftrightarrow \Sigma(TAT^{-1},CT^{-1}) \ \textit{ is observable} \end{aligned}$$

Theorem 51 ***A canonical form for uncontrollable system*** *Assume $\Sigma(A,B)$ is not controllable and define $k=\dim W<n$. Then, there exits a nonsigular $T\in\mathbb{R}^{n\times n}$ such that*

$$TAT^{-1}=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad TB=\begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with $A_{11}\in\mathbb{R}^{k\times k}$, $B_1\in\mathbb{R}^{k\times m}$ and $\Sigma(A_{11},B_1)$ controllable

Procedures:

1. Let $\{q_1,q_2,...,q_n\}$ with $q_i\in\mathbb{R}^n$ be a basis for $\mathbb{R}^n$ adapted to W, i.e $W=\operatorname{span}\{q_1,q_2,...,q_k\}$ and $\mathbb{R}^n=\operatorname{span}\{q_1,q_2,...,q_n\}$.
2. Define $T\in\mathbb{R}^{n\times n}$ such that $T^{-1}=\begin{bmatrix} q_1 & q_2 & \cdots & q_k & q_{k+1} & \cdots & q_n \end{bmatrix}$

Theorem 52 ***Canonical form for controllable system*** *Let $\Sigma(A,B)$ with $m=1$ be controllable. Then, there exists a nonsingular matrix $T\in\mathbb{R}^{n\times n}$ such that*

$$TAT^{-1}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad TB=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $a_0,...,a_{n-1}\in\mathbb{R}$ are the coefficients of the monic characteristic polynomial of A, i.e.,

$$\Delta_A(s)=s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-2}+\cdots+a_1s+a_0$$

Procedures:

1. Verify controllability
2. Compute $\Delta_A(s)$
3. Compute T

$$T^{-1} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

where

$$\begin{aligned} q_n &= B, \\ q_{n-1} &= AB + a_{n-1}B, \\ q_{n-2} &= A^2B + a_{n-1}AB + a_{n-2}B \\ &\vdots \\ q_1 &= A^{n-1}B + a_{n-1}A^{n-2}B + \cdots + a_2AB + a_1B \end{aligned}$$

Theorem 53 *Let $\Sigma(A,B)$ be such that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where $A_{11} \in \mathbb{R}^{k\times k}, B_1 \in \mathbb{R}^{m\times k}$ and $\Sigma(A_{11}, B_1)$ is controllable. Then, $\lambda \in \sigma(A)$ is (A,B) controllable if and only if $\lambda \notin \sigma(A_{22})$

Theorem 54 *The system $\Sigma(A,B)$*

$$\Sigma(A,B) : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \end{cases}$$

is controllable if and only if the system $\Sigma(A^T, B^T)$

$$\Sigma(A^T, B^T) : \begin{cases} \dot{x}(t) = A^T x(t) \\ y(t) = B^T x(t) \end{cases}$$

is observable.

Theorem 55 *Let $\Sigma(A,C)$ be unobservable and define $k = n - \dim \mathcal{N} < n$. Then, there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that*

$$TAT^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad CT^{-1} = \begin{bmatrix} C_1 & 0 \end{bmatrix} \tag{7}$$

where $A_{11} \in \mathbb{R}^{k\times k}$, $C_1 \in \mathbb{R}^{p\times k}$, and the matrix pair (A_{11}, C_1) is observable.

Controllable/observable eigenvalues

Definition 56 *An eigenvalue λ of $A \in \mathbb{R}^{n\times n}$ is called*

1. $(A,B)-$ *controllable if*
$$\operatorname{rank}[A-\lambda I \quad B] = n \tag{8}$$
2. $(A,C)-$ *observable if*
$$\operatorname{rank}\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = n$$

Remark: The rank conditions above allow for various alternative formulations. For example, instead of (8) we can write that for every vector v the implication

$$v^T A = \lambda v^T, \quad v^T B = 0 \Rightarrow v = 0$$

Thus, there does not exist a left eigenvector of A corresponding to the eigenvalue λ which is orthogonal to $\operatorname{im}(B)$. Similarly, (A,C) is unobservable if and only if $v \neq 0$ with

$$Av = \lambda v, \quad Cv = 0$$

i.e, v is a eigenvector which is in the null space (ker) of C.

Theorem 57 ***Hautus Test***

1. $\Sigma(A,B)$ *is controllable if and only if*
$$\operatorname{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n \quad \textit{for all } \lambda \in \sigma(A)$$
2. $\Sigma(A,C)$ *is observable if and only if*
$$\operatorname{rank}\begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = n \quad \textit{for all } \lambda \in \sigma(A)$$

Stabilization by static feedback

Let $\Sigma(A,B) : \dot{x}(t) = Ax(t) + Bu(t)$ where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$

Definition 58 ***State feedback controller***

$$u(t) = Fx(t)$$

Closed-loop dynamics

$$\dot{x}(t) = Ax(t) + BFx(t) = (A+BF)x(t)$$

Definition 59 ***Stabilization problem*** *Given $\Sigma(A,B)$, find $F \in \mathbb{R}^{m\times n}$ s.t*

$$\sigma(A+BF) \subset \mathbb{C}_- = \{z \in \mathbb{C} \mid \Re(z) < 0\}$$

Theorem 60 ***Pole placement*** *The following are equivalent*

1. $\Sigma(A, B)$ *is controllable*
2. *For every monic polynomial* p *of degree* n*, there exists* $F \in \mathbb{R}^{m\times n}$ *s.t*

$$\Delta_{A+BF}(s) = p(s)$$

Remark: hence, we can choose the eigenvalues of $A + BF$ anyway we like.

Theorem 61 *Given* $\Sigma(A, B)$*, there exist* $F \in \mathbb{R}^{m\times n}$ *such that* $\sigma(A + BF) \subset \mathbb{C}_-$ *if and only if every* $\lambda \in \sigma(A)$ *s.t* $\lambda \notin \mathbb{C}_-$ *is* (A, B) *controllable*

Definition 62 *The system* $\Sigma(A, B)$ *is called stabilizable if there exists a feedback* $F \in \mathbb{R}^{m\times n}$ *such that* $\sigma(A + BF) \subset \mathbb{C}_-$.

Corollary 63 *The system* $\Sigma(A, B)$ *is stabilizable if and only if every unstable eigenvalue of* A *is* (A, B)*-controllable, i.e., if and only if*

$$\operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \quad \textit{for all } \lambda \in \sigma(A) \quad \textit{s.t } \Re(\lambda) \geq 0$$

Theorem 64 *Consider the system* $\Sigma(A, B)$ *and assume that, for every initial condition* $x_0 \in \mathbb{R}^n$ *, there exists an input function* $u : [0, \infty) \to \mathbb{R}^m$ *such that*

$$\lim_{t\to\infty} x(t; x_0, u) = 0$$

Then, $\Sigma(A, B)$ *is stabilizable.*

Process to find the matrix F

1. First put the system Σ in canonical form
2. compute $\overline{F} = [f_0 \quad f_1 \quad \cdots \quad f_{n-1}]$ where $f_i = a_i - p_i$ for $i \in 0, ..., n-1$, with a_i and p_i the coefficient of the $\delta(A)$ and $\delta(A + BF)$ respectively. (a_0, p_0 are the coefficient of s^0 of both polynomial).
3. compute $F = \overline{F}T$

State Observers

Let the following system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(T) \\ y(t) = Cx(t) \end{cases}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$.
The goal of the observers Ω is to estimate the state of the real system Σ.

$$\Omega : \begin{cases} \dot{w}(t) = Pw(t) + Qu(t) + Ry(t) \\ \xi(t) = Sw(t) \end{cases}$$

with $w(t) \in \mathbb{R}^{n_w}$ and $\xi(t) \in \mathbb{R}^n$ an estimate of $x(t) \in \mathbb{R}^n$.

Definition 65 ***Estimation error***

$$e(t) = \xi(t) - x(t)$$

Dynamics:

$$\dot{e}(t) = (SP + SRCS - AS)w(t) + (A - SRC)e(t) + (SQ - B)u(t)$$

Definition 66 *Ω is called a state observer if, for any $x_0 \in \mathbb{R}^n$, $w_0 \in \mathbb{R}^{n_w}$ such that $e(0) = Sw_0 - x_0 = 0$ (i.e initially we have a perfect estimate), for any input function $u(\cdot)$,*

$$e(t) = 0, \quad \forall t \geq 0$$

Definition 67 *A state observer Ω is called stable if, for any $x_0 \in \mathbb{R}^n$, $w_0 \in \mathbb{R}^n$ and any input function $u(\cdot)$,*

$$\lim_{t \to \infty} e(t) = 0$$

Theorem 68 *The general form of a state observer for Σ is*

$$\dot{\xi}(t) = (A - SRC)\xi(t) + Bu(t) + SRy(t)$$

with $\xi(t) \in \mathbb{R}^n$, $SR = G \in \mathbb{R}^{n \times p}$. Then, the estimation error satisfies

$$\dot{e}(t) = (A - GC)e(t)$$

such that this state observer is stable if and only if $\sigma(A - GC) \subset \mathbb{C}_-$

Interpretation: output injection

$$\dot{\xi}(t) = A\xi(t) + Bu(t) + G(y(t) - C\xi(t))$$

therefore, the system Ω is a copy of the original system Σ, where ξ and $C\xi$ are the estimations for the state $x(\cdot)$ and output $y(\cdot)$ respectively.

Definition 69 *Σ is detectable if there exists $G \in \mathbb{R}^{n \times p}$ such that $\sigma(A - GC) \subset \mathbb{C}_-$*

Lemma 70 *The matrix pair (A, C) is detectable if and only if the matrix pair (A^T, C^T) is stabilizable*

Recall:

$$\begin{aligned} (A, B) \text{ is stabilizable } &\Leftrightarrow \exists F \text{ s.t. } \sigma(A + BF) \subset \mathbb{C}_- \\ (A, C) \text{ is detectable } &\Leftrightarrow \exists G \text{ s.t. } \sigma(A - GC) \subset \mathbb{C}_- \end{aligned}$$

Theorem 71 ***Hautus test for detectability*** *Σ is detectable if and only if every unstable eigenvalue of A is (A, C) observable, i.e.,*

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \sigma(A) \quad s.t\, \Re(\lambda) \geq 0$$

Corollary 72 *Consider Σ. The following are equivalent*

1. *There exists a stable state observer for Σ*
2. *Σ is detectable*
3. *every eigenvalue $\lambda \in \sigma(A)$ such that $\Re(\lambda) \geq 0$ is (A, C) observable*

stabilization by dynamic output feedback

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

with $x(t) \in \mathbb{R}^n$,$u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$

Definition 73 ***Dynamic output feedback controller***

$$\Gamma : \begin{cases} \dot{w}(t) = Kw(t) + Ly(t) \\ u(t) = Mw(t) + Ny(t) \end{cases}$$

with $w(t) \in \mathbb{R}^{n_w}$

Definition 74 ***Problem formulation*** *Given* $\Sigma(A, B, C)$. *Find* Γ *characterized by* $(K, L, M, N$ *such that*

$$A_{cl} = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}$$

satisfies $\sigma(A_{cl}) \subset \mathbb{C}_-$

Dynamic output feedback controller $(w = \xi)$

$$\Gamma : \begin{cases} \dot{\xi}(t) = (A - GC + BF)\xi(t) + Gy(t) \\ u(t) = F\xi(t) \end{cases}$$

Lemma 75 *Let*

1. Ω *be a stable observes for* Σ
2. F *solve the stabilization problem for static state feedback*

Then Γ *solves the stabilization problem by dynamic output feedback*

Theorem 76 *Consider* $\Sigma(A, B, C)$. *The following are equivalent:*

1. (A, B) *are stabilizable and* (A, C) *is detectable*
2. $\exists\Gamma$ *that solves the stabilization problem by dynamic output feedback*

$$\sigma(A_{cl}) \subset \mathbb{C}_- \quad \text{with } A_{cl} = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}$$

Input Output properties

Output Trajectory $(t_0 = 0)$

$$y(t; x_0, u) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\, d\tau \; + Du(t)$$

Definition 77 *Two system $\Sigma(A,B,C,D)$ and $\Sigma(\overline{A},\overline{B},\overline{C},\overline{D})$ are similar if*

$$\overline{A} = TAT^{-1}, \quad \overline{B} = TB, \quad \overline{C} = CT^{-1}, \quad \overline{D} = D$$

for some nonsingular T.

Theorem 78 *Let $\Sigma(A,B,C,D)$ and $\Sigma(\overline{A},\overline{B},\overline{C},\overline{D})$ be similar. Then,*

$$y(t;0,u) = \overline{y}(t;0,u)$$

for any input function $u : J \to \mathbb{R}^m$.

Input response

Definition 79 $u_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & -\epsilon \le t \le \epsilon \\ 0 & \textit{otherwise} \end{cases}$

Lemma 80 *Consider the system $\Sigma(A,B,C)$ and define*

$$y_\epsilon(t) = \int_{-\epsilon}^{t} Ce^{A(t-\tau)}Bu_\epsilon(\tau)\,d\tau$$

then

$$\lim_{\epsilon\to 0^+} y_\epsilon(t) = \begin{cases} Ce^{At}B, & t > 0 \\ 0, & t < 0 \end{cases}$$

Definition 81 ***Delta Dirac function*** *A function $\delta : \mathbb{R} \to \mathbb{R}$ with defining properties*

1. *$\delta(t) = 0$ for all $t \neq 0$*
2. *for any continuous function $\phi : \mathbb{R} \to \mathbb{R}$,*

$$\int_{-\infty}^{\infty} \phi(t-\tau)\delta(\tau)\,d\tau = \phi(t)$$

more explicitly

$$\int_{-\infty}^{\infty} \phi(t-\tau)\delta(\tau)\,d\tau = \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} \phi(t-\tau)u_\epsilon(\tau)\,d\tau =$$

take the input

$$u(t) = e_i\delta(t) \qquad \textit{with } e_i \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

where 1 is at position i. Then,

$$y(t;0,e_i\delta) = \int_0^t Ce^{A(t-\tau)}Be_i\delta(\tau)\,d\tau + De_i\delta(t) = \left(Ce^{At}B + D\delta(t)\right)e_i$$

Definition 82 *The impulse response matrix for Σ is $H : \mathbb{R} \to \mathbb{R}^{p\times m}$ defined as*

$$H(t) = \begin{cases} Ce^{At}B + D\delta(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Theorem 83 *Consider Σ with impulse response matrix H. Then,*

$$y(t; 0, u) = \int_0^t H(t-\tau)u(\tau)\, d\tau$$

Theorem 84 *Let the systems $\Sigma(A, B, C, D)$ and $\Sigma(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ be similar and denote their impulse response matrices by H and $\overline{H}$, respectively. Then,*

$$H(t) = \overline{H}(t)$$

for all $t \in \mathbb{R}$

Theorem 85 *Consider the system $\Sigma(A, B, C, D)$ and its impulse response matrix. Then, the following statements hold:*

1. *assume that the matrices $A, B,$ and C are structured as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

then the impulse response matrix satisfies

$$H(t) = \begin{cases} C_1 e^{A_{11}t}B_1 + D\delta(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

2. *assume that the matrices $A, B,$ and C are structured as*

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

then the impulse response matrix satisfies

$$H(t) = \begin{cases} C_1 e^{A_{11}t}B_1 + D\delta(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Remark: essentially states that the impulse response of a system is only dependent on its controllable subsystem.

Laplace transform

Definition 86 *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is exponentially bounded if*

$$|f(t)| \leq Me^{\alpha t} \quad \text{for all } t \in \mathbb{R}_+$$

Definition 87 *For f exponentially bounded, its Laplace transform*

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt$$

for $s \in \mathbb{C}$ with $\Re(s) > \alpha$

Theorem 88 ***Propreties***

1. $\mathcal{L}(f + f') = \mathcal{L}(f) + L(f')$, $\mathcal{L}(\alpha f) = \alpha\mathcal{L}(f)$, $\alpha \in \mathbb{C}$
2. *If f is diferentiable and $\dot{f}$ is exponentially bounded,*

$$\mathcal{L}(\dot{f}) = s\mathcal{L}(f) - f(0)$$

3. *if u, h are exponentially bounded, then*

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau$$

is exponentially bounded and

$$\mathcal{L}(y) = \mathcal{L}(h)\mathcal{L}(u)$$

Transfer function matrix

Denote $\hat{x}(s) = \mathcal{L}(x)(s)$, $\hat{u}(s) = \mathcal{L}(u)(s)$, $\hat{y}(s) = \mathcal{L}(y)(s)$

Definition 89 *The transfer function matrix of Σ is the function*

$$T(s) = C(sI - A)^{-1}B + D$$

Theorem 90 *Consider the system Σ with H and T. Then,*

$$T(s) = \mathcal{L}(H)(s)$$

for all $s \in \mathbb{C}$ such that $\Re(s) > \Lambda(A)$ where $\Lambda(A) = \max\{\Re(\lambda) | \lambda \in \sigma(A)\}$

Theorem 91 *Let $\Sigma(A, B, C, D)$ and $\Sigma(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ be similar. Then,*

$$T(s) = \overline{T}(s)$$

Theorem 92 ***Cramer's rule***

$$(sI - A)^{-1} = \frac{1}{\Delta_A(s)} adj(sI - A)$$

SISO systems

SISO means Single Input Single Output

$$\Sigma_{siso} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

where $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$.

Theorem 93 *Consider Σ_{siso} with*

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \end{bmatrix}$$

where $D = 0$. Then,

$$T(s) = \frac{c_{n-1}s^{n-1} + c_{n-2}s^{n-2} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

Theorem 94 *Consider Σ_{siso} and let (A, B) be controllable. Then,*

$$p(s) = Cadj(sI - A)B, q(s) = \Delta_A(s)$$

are coprime (i.e they do not have common roots) if and only if (A, C) is observable.

Stability

Definition 95 *$\lambda \in \mathbb{C}$ is a pole of T if it is a root of q', where q', p' are the coprime polynomials of T. It is a pole of*

$$T(s) = \frac{1}{q(s)} P(s)$$

with P a matrix of polynomials, if it is a pole of at least one of its elements.

Theorem 96 *1. If $\lambda \in \mathbb{C}$ is a pole of T, then $\lambda \in \sigma(A)$.*

2. if $\lambda \in \sigma(A)$, (A, B) is controllable, (A, C) is observable, then λ is a pole of T.

Definition 97 *Σ is externally stable if $\exists \gamma > 0$ s.t for any bounded $u : \mathbb{R}_+ \to \mathbb{R}^m$,*

$$\sup_{t \in \mathbb{R}_+} \|y(t; 0, u)\| \leq \gamma \sup_{t \in \mathbb{R}_+} \|u(t)\|$$

Remark: Due to linearity

$$\|u(t)\| \leq 1 \quad \forall t \in \mathbb{R}_+ \Rightarrow \|y(t; 0, u)\| \leq \gamma \quad \forall t \in \mathbb{R}_+$$

Lemma 98 *$\Sigma(A, B, C, D)$ is externally stable if and only if $\Sigma(A, B, C, 0)$ is externally stable*

Theorem 99 *The following are equivalent:*

1. *Σ is externally stable*
2. *$\int_0^\infty \|Ce^{At}B\| \, dt < \infty$*
3. *$\lim_{t\to\infty} Ce^{At}B = 0$*
4. *all poles of T are in $\mathbb{C}_-$*

Definition 100 *Σ is internally stable if $\dot{x}(t) = Ax(t)$ is asymptotically stable, i.e*

$$\lim_{t\to\infty} e^{At} = 0$$

Theorem 101

1. *If Σ is internally stable, then Σ is externally stable*
2. *If Σ is external stable, (A, B) controllable, (A, C) observable, then Σ is internally stable*